Malaysian Journal of Mathematical Sciences 19(4): 1453–1470 (2025)
https://doi.org/10.47836/mjms.19.4.15

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Computation of Distance-Based Polynomials and Topological Indices for Generalized Friendship Graphs ($\mathscr{G}(n,k)$): A Python-Based Approach

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Received: 11 March 2025
Accepted: 13 June 2025

Abstract

In this research, we present a Python-based algorithm for efficiently computing distance-based topological indices and polynomials for the generalized friendship graph $\mathscr{G}(n,k)$. We calculate both Hosoya polynomial and Harary polynomial and establish indices which include the Wiener index, hyper-Wiener index, Harary index, and multiplicative Wiener index. The paper discusses new index modifications which include modified Wiener index combined with modified hyper-Wiener index and modified Harary index and their subsequent analysis. The proposed Python algorithm uses the Floyd-Warshall method to efficiently compute distances between graph nodes, enabling the rapid determination of all relevant indices and associated polynomials. This paper presents visualization graphs to demonstrate the changes observed in the indices as graph parameters vary. We use computational methods to develop knowledge about graph polynomials and indices resulting in practical application possibilities for modeling social networks and theoretical graph examination.

Keywords: distance; generalized friendship graph; Hosoya polynomial; Harary polynomial; Wiener index; hyper Wiener index; Harary index; multiplicative Wiener index; Python algorithm.

1 Introduction

Graph theory, a vital area of mathematics, plays a crucial role in numerous scientific and engineering fields. A graph $G(V, E)$ works as an elementary structure which unites V nodes with E links between them [22]. The total number of edges linking to a single vertex creates its degree value and the shortest edge-based connection route between two vertices is known as distance and it is denoted by $d(u_1, u_2)$. The generalized friendship graph $\mathscr{G}(n, k)$ having a central node and multiple disjoint cliques structure gives this graph an exceptional framework for developing studies about distance-based topological indices and fundamental polynomials in graph theory. The connectivity and operational efficiency of graphs become clear through the measurements that are provided by Wiener index along with Harary index and the hyper-Wiener index as distance-based indices. These structural properties play a central role in network behavior, so scientists use these indices for their widespread adoption as quantification tools. The Hosoya polynomial together with the Harary polynomial represent two types of graph polynomials that provide mathematical treatments of graph distance information for a better understanding of graph topology [14]. Although distance-based topological polynomials and indices have been investigated for friendship graphs [8], their study on generalized friendship graphs $\mathscr{G}(n, k)$ remains insufficiently explored in current research.

We calculate both Hosoya polynomial and Harary polynomial alongside known indices which include Wiener index and the hyper-Wiener index, the Harary index as well as the multiplicative Wiener index. The assessment of structural details in $\mathscr{G}(n, k)$ requires the introduction of the modified Wiener index, the modified hyper-Wiener index and the modified Harary index methods. A Python algorithm helps with calculating graph distances while evaluating both indices and polynomials to enable effective measurement computation. A series of visual demonstrations shows the performance of these indices when the parameters of the graphs change. Analysis tools which are developed from our study contribute in both theoretical value to understand $\mathscr{G}(n, k)$ and practical evaluation that capabilities for network design applications and social network modeling.

1.1 Literature review

The Wiener index stands as an essential topological descriptor within graph theory because it has received extensive examination regarding its chemical and mathematical applications. Dobrynin et al. [6] establish a thorough theoretical model for the Wiener index which was applied to trees as it helps researchers to analyze molecular structures. According to Bonchev [4], the Wiener number receives additional consideration regarding its chemical graph theory applications and the latest advancements within the field. The Harary index stands as an additional significant topological index such that experts apply it in chemical graph theory studies. Yu et al. [15] conducted a detailed study of the Harary index for trees which explores both computational aspects as well as theoretical implications.

The generalized friendship graph has been studied by researchers in graph theory with a focus on understanding its structural properties and potential applications. Yin et al. [24] studied graphic sequences that create generalized friendship graphs to deliver essential knowledge regarding their structural properties. Arumugam and Nalliah [2] conducted research on super $(a, d)-$edge antimagic total labelings through their study of these graph types. The research by Gao et al. [8] focuses on assessing distance-based topological polynomials and indices to establish their value for measuring graph properties. The research conducted by Cahyabudi and Kusmayadi [5] examines local metric dimensions within friendship graphs for their importance to graph metric theory. The paper by Baca et al. [3] introduces edge irregular reflexive labelings as a fresh approach to understand graph labeling approaches. Nalliah et al. [17] showed how generalized friendship graphs remain significant for combinatorial graph theory on local antimagic vertex coloring in their latest publication. These works demonstrate the extensive theoretical value and wide application potential of generalized friendship graphs as a whole.

Chemical graph theory research defines Hosoya and Harary polynomials as significant subjects for evaluating molecular structures. Liu et al. [16] investigate Zagreb and multiplicative Zagreb indices for Eulerian graphs, providing theoretical bounds and computational results. Ahmad et al. [1] conducted research on distance-based invariants of zigzag polyhex nanotubes as these structures matter in nanotechnology and material science. The paper by Wang et al. [21] demonstrated the capability of Hosoya and Harary polynomials for modeling intricate nanostructures by applying them to TUC4 nanotubes. These studies show the vital role in which Hosoya and Harary polynomials contribute to establish molecular and nanomaterial properties. Gowtham and Husin [9] analyze reverse topological indices for families of bistar and corona product graphs, contributing to the study of chemical graph theory and network descriptors. Havare [13] analyzes reformulated Zagreb indices for cycle-related graphs and linear [n]-phenylenes, contributing to chemical graph theory. Semil @ Ismail et al. [20] compute distance-based topological indices for zero-divisor graphs of commutative rings, linking algebra and graph theory.

1.2 The importance of generalized friendship graphs

A generalized friendship graph $\mathscr{G}(n,k)$ is crucial for graph theory due to its unique structural characteristics and multipurpose representation of the relational structure. Distance-based topological indices and polynomials are commonly used to analyze the structural properties of networks [6]. In this context, the generalized friendship graph has been examined for its potential relevance in modeling and analyzing social network structures [7]. Each social network contains complex interconnections of users who become nodes while resulting relationships become edges. Distance-based indices of generalized friendship graphs provide an analytical framework to understand social relationships through the calculation of individual closeness and separation measures. The Wiener index quantifies the efficiency of information distribution in a network by summing the shortest path distances between all pairs of nodes [4]. In contrast, the Harary index measures network compactness using the sum of reciprocals of these distances, thereby emphasizing the accessibility of nodes and potential interaction efficiency [18]. Complex networks can be effectively analyzed through distance-based topological indices and polynomials using the generalized friendship graph. Additionally, distance-based indices have been shown to capture the structural features of social, biological, and communication networks, offering insights into their connectivity and irregularities [11].

The research thoroughly examines the mathematical indices by describing their operational strengths and weaknesses relative to chemical network types. The calculations of distance-based topological polynomials and indices are provided in Table 1.

Table 1: Distance based indices.

Indices	**Notations**	**Formula**
Hosoya Polynomial [14]	$H(G,x)$	$\frac{1}{2}\sum_{u_1,u_2\in V(G)} x^{d(u_1,u_2)}.$
Harary Polynomial [12]	$h(G,x)$	$\frac{1}{2}\sum_{u_1,u_2\in V(G)} \frac{1}{d(u_1,u_2)}x^{d(u_1,u_2)}.$
Wiener Index [23]	$W(G)$	$\frac{1}{2}\sum_{u_1,u_2\in V(G)} d(u_1,u_2).$
Hyper-Wiener Index [10]	$WW(G)$	$\frac{1}{2}\sum_{u_1,u_2\in V(G)} \big(d(u_1,u_2)+d(u_1,u_2)^2\big).$
Multiplicative Wiener Index [19]	$\pi(G)$	$\frac{1}{2}\prod_{u_1,u_2\in V(G)} d(u_1,u_2).$
Harary Index [18]	$H(G)$	$\frac{1}{2}\sum_{u_1,u_2\in V(G)} \frac{1}{d(u_1,u_2)}.$

2 Methodology

The proposed methodology begins with creating the generalized friendship graph $\mathscr{G}(n,k)$ which consists of k cycles. These cycles all share a common central vertex. Each of those k cyclic components uses the same central vertex as its base point. The construction of the distance matrix D proceeds while D_{ij} represents the shortest path distance between vertices i and j. Algorithms Dijkstra's serve as the tool for this purpose. The analysis of distances includes identifying the cardinality by counting each pair of vertices at distinct distance values. The study presents a summary including findings that demonstrate the connection between variable values n, k and distance measurements. The investigation divides the results into two sections according to the parity of n when determining whether n is an even or odd integer. Defining distance patterns becomes possible through this approach. The distance matrix creates distance-based polynomials starting from the Hosoya polynomial through the Harary polynomial. The polynomials provide a mathematical representation of all distance properties which are found in the graph.

Distance-based indices serve to evaluate the topological features of the graph. Indeed, the Wiener index,the hyper-Wiener index and the multiplicative Wiener index and Harary index provide individual structural insights about the graph's structure. These indices provide measurements that determine structural factors like the amount of compactness in addition to path influence and distance multipliers. The analysis examines every aspect of $\mathscr{G}(n,k)$ to provide in-depth knowledge about its structure as well as its topological characteristics. The workflow is illustrated in the methodology flowchart presented in Figure 1.

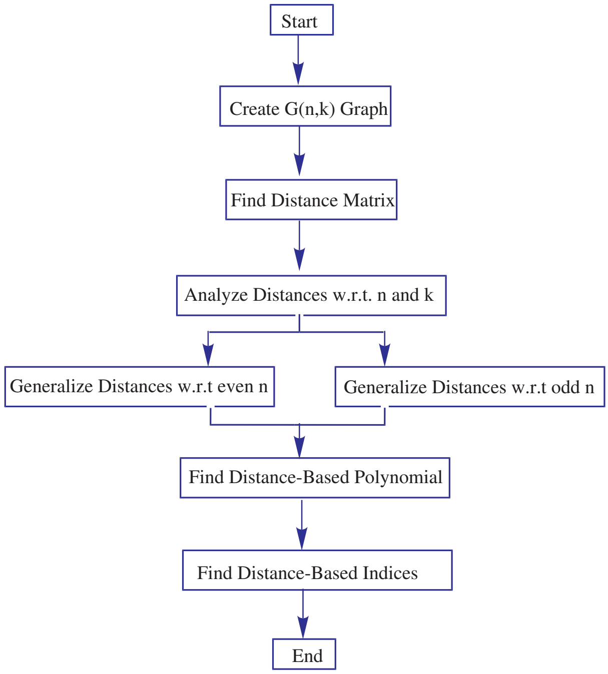

Figure 1: A flow chart.

3 Results and Discussion

The distance based polynomial calculation is performed through the distance partitioning method. The result validation process utilized Python programming language. Analyses have been performed with the information presented in Tables 2 and 3.

3.1 Generalized friendship graphs

A generalized friendship graph ($\mathscr{G}(n,k)$ or simply $\mathscr{G}$) features a central vertex (or hub) connected to multiple disjoint cycles or cliques, with each cycle or clique sharing only the central vertex. Here, n represents the size of each cycle, and k denotes the number of cycles. This structure extends the idea of a friendship graph, where triangles share a common central vertex, as illustrated in Figure 2.

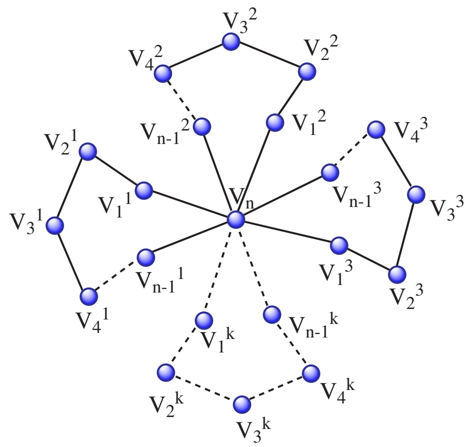

Figure 2: Generalized friendship graph $\mathscr{G}(n, k)$.

Figure 3 displays the adjacency matrix for $n = k = 10$, while Figure 4 shows the corresponding distance matrix for $n = k = 10$.

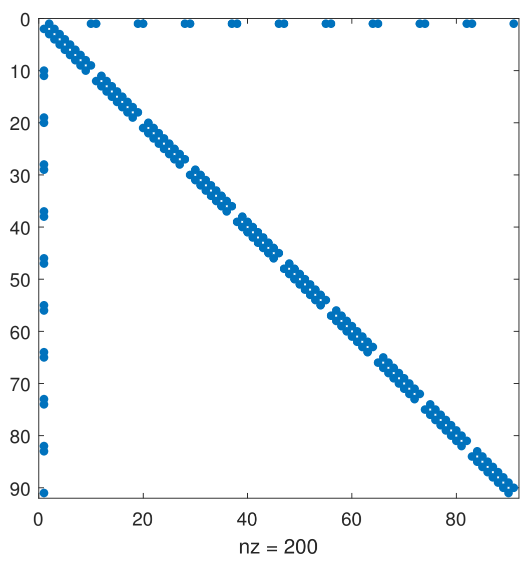

Figure 3: Adjacency matrix of generalized friendship graph for $n = k = 10$.

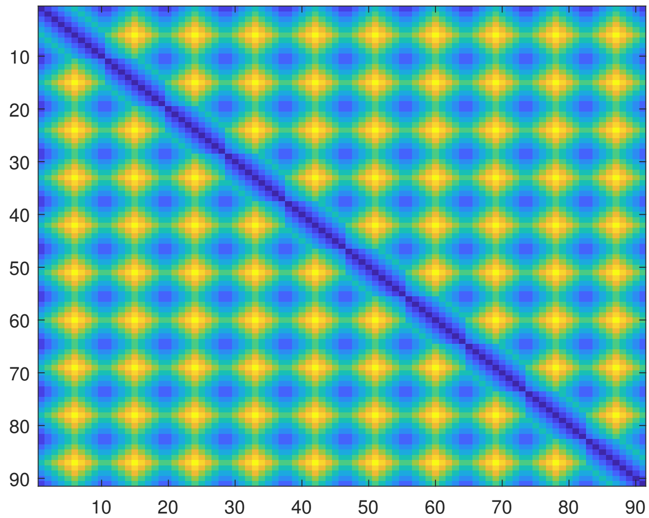

Figure 4: Distance matrix of generalized friendship graph for $n = k = 10$.

Lemma 3.1. *The graph $\mathscr{G}$ has $nk - k + 1$ vertices and nk edges.*

Theorem 3.1. *Let $\mathscr{G}$ be a generalized friendship graph, where $k \geq 1$ and $n \equiv 1 \mod 2$. Then, the distance based polynomial of $\mathscr{G}$ is given as follows,*

$$H(\mathscr{G}; x) = nk - k + 1 + nkx + \sum_{d=2, n>3}^{\frac{n-1}{2}} \left((n - 2d + 2)k + (2d - 2)k^2\right) x^d$$
$$+ \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d - 2n)k + (2n - 2d)k^2\right) x^d.$$

Proof. The term $nk - k + 1$ represents the number of pairs of vertices at distance 0, which is simply the number of vertices in the graph. The term nkx accounts for the edges in the graph (each cycle contributes n edges, and there are k cycles). Each edge corresponds to a pair of vertices at distance 1.

For $d \geq 2$, the sum from $d = 2$ to $d = \dfrac{n-1}{2}$ counts pairs of vertices at distance d. These pairs can be within the same cycle (C_n) or between different cycles, with distances summing to d. The coefficient $(n - 2d + 2)k + (2d - 2)k^2$ is derived from the number of such pairs.

For larger distances d, the sum from $d = \dfrac{n+1}{2}$ to $d = n - 1$ is adjusted to account for the cyclic structure of the graph. The coefficient $(2d - 2n)k + (2n - 2d)k^2$ reflects the number of pairs at these distances. By combining the above analysis and the Hosoya polynomial for the generalized friendship graph $\mathscr{G}$ we get,

$$H(\mathscr{G};x) = nk - k + 1 + nkx + \sum_{d=2,n>3}^{\frac{n-1}{2}} \left((n-2d+2)k + (2d-2)k^2\right) x^d$$
$$+ \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d-2n)k + (2n-2d)k^2\right) x^d.$$

This polynomial captures the distances between all any pairs of vertices in $\mathscr{G}$, as derived from Figure 2 of the graph and the generalized distance in Table 2.

Table 2: Generalized distances of $\mathscr{G}$ for $n \equiv 1 \mod 2$.

D\n	3	5	7	9	11	. . .	n
1	$3k$	$5k$	$7k$	$9k$	$11k$	. . .	nk
2	$2k^2-2k$	$2k^2+3k$	$2k^2+5k$	$2k^2+7k$	$2k^2+9k$	. . .	$2k^2+(n-2)k$
3		$4k^2-4k$	$4k^2+3k$	$4k^2+5k$	$4k^2+7k$	. . .	$4k^2+(n-4)k$
4		$2k^2-2k$	$6k^2-6k$	$6k^2+3k$	$6k^2+5k$	. . .	$6k^2+(n-6)k$
.			$4k^2-4k$	$8k^2-8k$	$8k^2+3k$	. . .	.
.			$2k^2-2k$	$6k^2-6k$	$10k^2-10k$	. . .	.
.				$4k^2-4k$	$8k^2-8k$	. . .	.
(n-1)/2				$2k^2-2k$	$6k^2-6k$	. . .	$(n-3)k^2+(n-(n-3))k$
(n+1)/2					$4k^2-4k$	. . .	$(n-1)k^2+(1-n)k$
(n+3)/2					$2k^2-2k$	. . .	$(n-3)k^2+(3-n)k$
(n+5)/2						. . .	$(n-5)k^2+(5-n)k$
(n+7)/2						. . .	.
(n+9)/2						. . .	.
.						. . .	.
.						. . .	$6k^2-6k$
.							$4k^2-4k$
n-1							$2k^2-2k$

□

Theorem 3.2. *Let $\mathscr{G}$ be a generalized friendship graph, where $k \geq 1$ and $n \equiv 1 \mod 2$. Then, the distance based polynomial of $\mathscr{G}$ is*

$$h(\mathscr{G};x) = +nkx + \sum_{d=2,n>3}^{\frac{n-1}{2}} \frac{\left((n-2d+2)k + (2d-2)k^2\right)}{d} x^d + \sum_{d=\frac{n+1}{2}}^{n-1} \frac{\left((2d-2n)k + (2n-2d)k^2\right)}{d} x^d.$$

Proof. The proof is straightforward and can be directly derived from the Hosoya polynomial by incorporating the reciprocal of the distances provided in Theorem 3.1. □

Theorem 3.3. *Let $\mathscr{G}$ be a generalized friendship graph, where $k \geq 1$ and $n \equiv 1$ mod 2. Then, the distance based indices of $\mathscr{G}$ are*

$$W_\lambda(\mathscr{G}) = nk + \sum_{d=2,n>3}^{\frac{n-1}{2}} \left((n-2d+2)k + (2d-2)k^2\right) d^\lambda + \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d-2n)k + (2n-2d)k^2\right) d^\lambda,$$

$$\begin{aligned}
WW_\lambda(\mathscr{G}) &= 2nk + \sum_{d=2,n>3}^{\frac{n-1}{2}} \left((n-2d+2)k + (2d-2)k^2\right)\left(d^\lambda + d^{2\lambda}\right) \\
&\quad + \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d-2n)k + (2n-2d)k^2\right)\left(d^\lambda + d^{2\lambda}\right), \\
h_t(\mathscr{G}) &= \frac{nk}{1+t} + \sum_{d=2,n>3}^{\frac{n-1}{2}} \frac{\left((n-2d+2)k + (2d-2)k^2\right)}{d+t} + \sum_{d=\frac{n+1}{2}}^{n-1} \frac{\left((2d-2n)k + (2n-2d)k^2\right)}{d+t}, \\
\pi(\mathscr{G}) &= 1 + \sum_{d=2,n>3}^{\frac{n-1}{2}} d^{\left((n-2d+2)k+(2d-2)k^2\right)} + \sum_{d=\frac{n+1}{2}}^{n-1} d^{\left((2d-2n)k+(2n-2d)k^2\right)}.
\end{aligned}$$

Proof. For the modified Wiener index, we have

$$\begin{aligned}
W_\lambda(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} d(u_1,u_2)^\lambda \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} 1^\lambda + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} d^\lambda \\
&= nk + \sum_{d=2,n>3}^{\frac{n-1}{2}} \left((n-2d+2)k + (2d-2)k^2\right) d^\lambda + \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d-2n)k + (2n-2d)k^2\right) d^\lambda.
\end{aligned}$$

For the modified hyper Wiener index, we have

$$\begin{aligned}
W_\lambda(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(d(u_1,u_2)^\lambda + d(u_1,u_2)^{2\lambda}\right) \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(1^\lambda + 1^{2\lambda}\right) + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(d^\lambda + d^{2\lambda}\right) \\
&= 2nk + \sum_{d=2,n>3}^{\frac{n-1}{2}} \left((n-2d+2)k + (2d-2)k^2\right)\left(d^\lambda + d^{2\lambda}\right) \\
&\quad + \sum_{d=\frac{n+1}{2}}^{n-1} \left((2d-2n)k + (2n-2d)k^2\right)\left(d^\lambda + d^{2\lambda}\right).
\end{aligned}$$

For the generalized Harary index, we have

$$\begin{aligned}
h_t(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{d(u_1,u_2)+t} \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{1+t} + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{d+t} \\
&= \frac{nk}{1+t} + \sum_{d=2,n>3}^{\frac{n-1}{2}} \frac{\left((n-2d+2)k + (2d-2)k^2\right)}{d+t} + \sum_{d=\frac{n+1}{2}}^{n-1} \frac{\left((2d-2n)k + (2n-2d)k^2\right)}{d+t}.
\end{aligned}$$

For the multiplicative Wiener index, we have

$$\begin{aligned}\pi(\mathscr{G}) &= \prod_{u_1,u_2\in(\mathscr{G})} d(u_1,u_2)\\ &= \prod_{u_1,u_2\in(\mathscr{G})} (1) \times \prod_{u_1,u_2\in(\mathscr{G})} (d)\\ &= 1 \times \prod_{d=2,n>3}^{\frac{n-1}{2}} d^{\left((n-2d+2)k+(2d-2)k^2\right)} \times \prod_{d=\frac{n+1}{2}}^{n-1} d^{\left((2d-2n)k+(2n-2d)k^2\right)}.\end{aligned}$$

$\square$

Corollary 3.1. *The Wiener index of $\mathscr{G}$ is given as follows;*

Case 1: *For $n = 3$,*

$$W_1(\mathscr{G}(3,k)) = k(-1+4k).$$

Case 2: *For $n > 3$,*

$$W_1(\mathscr{G}) = \frac{1}{4}k^2(n+1)(n-1)^2 + k\left(-\frac{1}{8}n((n-2)n+7) - \frac{1}{4}\right) + nk.$$

Corollary 3.2. *The hyper Wiener index of $\mathscr{G}$ is given as follows;*

Case 1: *For $n = 3$,*

$$WW_1(\mathscr{G}(3,k)) = -6k + 12k^2.$$

Case 2: *For $n > 3$,*

$$\begin{aligned}WW_1(\mathscr{G}) = {}& \frac{1}{48}k^2(n+1)(7n+15)(n-1)^2 + \frac{1}{48}k(-n(n(n(5n+2)-20)+94)-15)\\ &+ 2nk.\end{aligned}$$

Corollary 3.3. *The Harary index of $\mathscr{G}$ is given as follows,*

$$\begin{aligned}h_0(\mathscr{G}) = nk &+ \sum_{d=2,n>3}^{\frac{n-1}{2}} \frac{\left((n-2d+2)k+(2d-2)k^2\right)}{d}\\ &+ \sum_{d=\frac{n+1}{2}}^{n-1} \frac{\left((2d-2n)k+(2n-2d)k^2\right)}{d}.\end{aligned}$$

Theorem 3.4. *Let $\mathscr{G}$ be a generalized friendship graph, where $k \geq 1$ and $n \equiv 0 \bmod 2$. Then, the distance based polynomial of $\mathscr{G}$ is*

$$\begin{aligned}H(\mathscr{G};x) = {}& nk - k + 1 + nkx + \sum_{d=2,n>4}^{\frac{n-2}{2}} \left((n-2d+2)k+(2d-2)k^2\right)x^d\\ &+ \left(\frac{(2-n)k}{2} + (n-2)k^2\right)x^{\frac{n}{2}}\\ &+ \sum_{d=\frac{n+2}{2}}^{n-1} \left((2d-2n)k+(2n-2d)k^2\right)x^d + \left(\frac{-k}{2}+\frac{k^2}{2}\right)x^n.\end{aligned}$$

Proof. The term $nk-k+1$ represents the number of pairs of vertices at distance 0, which is simply the number of vertices in the graph. The term nk accounts for the edges in the graph (each cycle contributes n edges, and there are k cycles). Each edge corresponds to a pair of vertices at distance 1.

For $d \geq 2$, sum from $d=2$ to $d=\dfrac{n-2}{2}$ counts pairs of vertices at distance d. These pairs can be within the same cycle and between different cycles, with distances summing to d. The coefficient $(n-2d+2)k+(2d-2)k^2$ is derived from the number of such pairs, as shown in the Table 3. For $d=\dfrac{n}{2}$, the coefficient $\dfrac{(2-n)k}{2}+(n-2)k^2$ is derived from the Table 3. This term accounts for pairs of vertices at the maximum distance within a cycle.

For larger distance d, sum over $d=\dfrac{n+2}{2}$ to $d=n-1$ the counting is adjusted to account for the cyclic structure of the graph. The coefficient $(2d-2n)k+(2n-2d)k^2$ reflects the number of pairs at these distances, as shown in the Table 3. For $d=n$, the coefficient $\dfrac{-k}{2}+\dfrac{k^2}{2}$ is derived from Table 3. This term accounts for pairs of vertices at the maximum possible distance in the graph.By combining the above analysis with the Hosoya polynomial for the generalized friendship graph $\mathscr{G}$, we get

$$\begin{aligned} H(\mathscr{G};x) = nk-k+1+nkx &+ \sum_{d=2, n>4}^{\frac{n-2}{2}} \left((n-2d+2)k+(2d-2)k^2\right)x^d \\ &+ \left(\frac{(2-n)k}{2}+(n-2)k^2\right)x^{\frac{n}{2}} \\ &+ \sum_{d=\frac{n+2}{2}}^{n-1} \left((2d-2n)k+(2n-2d)k^2\right)x^d + \left(\frac{-k}{2}+\frac{k^2}{2}\right)x^n. \end{aligned}$$

This polynomial captures the distances between all pairs of vertices in $\mathscr{G}$, as derived from the structure depicted in Figure 2 and the generalized distance analysis provided in Table 3.

Table 3: Generalized distances of $\mathscr{G}$ for $n \equiv 0 \bmod 2$.

D\n	4	6	8	10	12	. . .	n
1	$4k$	$6k$	$8k$	$10k$	$12k$	. . .	nk
2	$2k^2+0k$	$2k^2+4k$	$2k^2+6k$	$2k^2+8k$	$2k^2+10k$	. . .	$2k^2+(n-2)k$
3	$2k^2-2k$	$4k^2-k$	$4k^2+4k$	$4k^2+6k$	$4k^2+8k$	. . .	$4k^2+(n-4)k$
4	$\frac{1}{2}k^2-\frac{1}{2}k$	$4k^2-4k$	$6k^2-2k$	$6k^2+4k$	$6k^2+6k$	. . .	$6k^2+(n-6)k$
.		$2k^2-2k$	$6k^2-6k$	$8k^2-3k$	$8k^2+4k$	. . .	.
.		$\frac{1}{2}k^2-\frac{1}{2}k$	$4k^2-4k$	$8k^2-8k$	$10k^2-4k$	. . .	.
.			$2k^2-2k$	$6k^2-6k$	$10k^2-10k$	. . .	.
(n-2)/2			$\frac{1}{2}k^2-\frac{1}{2}k$	$4k^2-4k$	$8k^2-8k$	. . .	$(n-3)k^2+3k$
(n)/2				$2k^2-2k$	$6k^2-6k$	. . .	$(n-2)k^2+\frac{(4-n)k}{2}$
(n+2)/2				$\frac{1}{2}k^2-\frac{1}{2}k$	$4k^2-4k$	. . .	$(n-2)k^2+(2-n)k$
(n+4)/2					$2k^2-2k$	. . .	$(n-4)k^2+(4-n)k$
(n+6)/2					$\frac{1}{2}k^2-\frac{1}{2}k$	. . .	$(n-6)k^2+(6-n)k$
(n+8)/2						. . .	.
.						. . .	.
.						. . .	.
.						. . .	$6k^2-6k$
.						. . .	$4k^2-4k$
n-1						. . .	$2k^2-2k$
n							$\frac{1}{2}k^2-\frac{1}{2}k$

□

Theorem 3.5. *Let $\mathscr{G}$ be a generalized friendship graph, where $k \geq 1$ and $n \equiv 0 \mod 2$. Then, the distance based polynomial of $\mathscr{G}$ is given as follows,*

$$h(\mathscr{G};x) = nkx + \sum_{d=2,n>4}^{\frac{n-2}{2}} \frac{\left((n-2d+2)k+(2d-2)k^2\right)}{d} x^d + \frac{\left(\frac{(2-n)k}{2}+(n-2)k^2\right)}{\frac{n}{2}} x^{\frac{n}{2}} + \sum_{d=\frac{n+2}{2}}^{n-1} \frac{\left((2d-2n)k+(2n-2d)k^2\right)}{d} x^d + \frac{\left(\frac{-k}{2}+\frac{k^2}{2}\right)}{d} x^n.$$

Proof. The proof is straightforward and can be directly derived from the Hosoya polynomial by incorporating the reciprocal of the distances that are given in Theorem 3.4. □

Theorem 3.6. *Let $\mathscr{G}$ be the generalized friendship graph , where $k \geq 1$ and $n \equiv 0 \mod 2$. Then, the distance based indices of $\mathscr{G}$ are*

$$W_\lambda(\mathscr{G}) = nk + \sum_{d=2,n>4}^{\frac{n-2}{2}} \left((n-2d+2)k+(2d-2)k^2\right) d^\lambda + \left(\frac{(2-n)k}{2}+(n-2)k^2\right)\left(\frac{n}{2}\right)^\lambda + \sum_{d=\frac{n+2}{2}}^{n-1} \left((2d-2n)k+(2n-2d)k^2\right) d^\lambda + \left(\frac{-k}{2}+\frac{k^2}{2}\right) n^\lambda,$$

$$WW_\lambda(\mathscr{G}) = 2nk + \sum_{d=2,n>4}^{\frac{n-2}{2}} \left((n-2d+2)k+(2d-2)k^2\right)(d^\lambda+d^{2\lambda}) + \left(\frac{(2-n)k}{2}+(n-2)k^2\right)\left(\left(\frac{n}{2}\right)^\lambda+\left(\frac{n}{2}\right)^{2\lambda}\right) + \sum_{d=\frac{n+2}{2}}^{n-1} \left((2d-2n)k+(2n-2d)k^2\right)(d^\lambda+d^{2\lambda}) + \left(\frac{-k}{2}+\frac{k^2}{2}\right)(n^\lambda+n^{2\lambda}),$$

$$h_t(\mathscr{G}) = \frac{nk}{1+t} + \sum_{d=2,n>4}^{\frac{n-2}{2}} \frac{\left((n-2d+2)k+(2d-2)k^2\right)}{d+t} + \frac{\left(\frac{(2-n)k}{2}+(n-2)k^2\right)}{\frac{n}{2}+t} + \sum_{d=\frac{n+2}{2}}^{n-1} \frac{\left((2d-2n)k+(2n-2d)k^2\right)}{d+t} + \frac{\left(\frac{-k}{2}+\frac{k^2}{2}\right)}{n+t},$$

$$\pi(\mathscr{G}) = 1 + \sum_{d=2,n>4}^{\frac{n-2}{2}} d^{\left((n-2d+2)k+(2d-2)k^2\right)} + \left(\frac{n}{2}\right)^{\left(\frac{(2-n)k}{2}+(n-2)k^2\right)} + \sum_{d=\frac{n+2}{2}}^{n-1} d^{\left((2d-2n)k+(2n-2d)k^2\right)} + n^{\left(\frac{-k}{2}+\frac{k^2}{2}\right)}.$$

Proof. For the modified Wiener index, we have

$$
\begin{aligned}
W_\lambda(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} d(u_1,u_2)^\lambda \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} 1^\lambda + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} d^\lambda \\
&= nk + \sum_{d=2,n>4}^{\frac{n-2}{2}} \left((n-2d+2)k+(2d-2)k^2\right) d^\lambda + \left(\frac{(2-n)k}{2}+(n-2)k^2\right)\left(\frac{n}{2}\right)^\lambda \\
&\quad + \sum_{d=\frac{n+2}{2}}^{n-1} \left((2d-2n)k+(2n-2d)k^2\right) d^\lambda + \left(\frac{-k}{2}+\frac{k^2}{2}\right) n^\lambda.
\end{aligned}
$$

For the modified Hyper Wiener index, we have

$$
\begin{aligned}
W_\lambda(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(d(u_1,u_2)^\lambda + d(u_1,u_2)^{2\lambda}\right) \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(1^\lambda+1^{2\lambda}\right) + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \left(d^\lambda+d^{2\lambda}\right).
\end{aligned}
$$

For the generalized Harary index, we have

$$
\begin{aligned}
h_t(\mathscr{G}) &= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{d(u_1,u_2)+t} \\
&= \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{1+t} + \frac{1}{2}\sum_{u_1,u_2\in\mathscr{G}} \frac{1}{d+t} \\
&= \frac{nk}{1+t} + \sum_{d=2,n>4}^{\frac{n-2}{2}} \frac{\left((n-2d+2)k+(2d-2)k^2\right)}{d+t} + \frac{\left(\frac{(2-n)k}{2}+(n-2)k^2\right)}{\frac{n}{2}+t} \\
&\quad + \sum_{d=\frac{n+2}{2}}^{n-1} \frac{\left((2d-2n)k+(2n-2d)k^2\right)}{d+t} + \frac{\left(\frac{-k}{2}+\frac{k^2}{2}\right)}{n+t}.
\end{aligned}
$$

For the multiplicative Wiener index, we have

$$
\begin{aligned}
\pi(\mathscr{G}) &= \prod_{u_1,u_2\in(\mathscr{G})} d(u_1,u_2) \\
&= \prod_{u_1,u_2\in(\mathscr{G})} (1) \times \prod_{u_1,u_2\in(\mathscr{G})} (d) \\
&= 1 \times \prod_{d=2,n>4}^{\frac{n-2}{2}} d^{\left((n-2d+2)k+(2d-2)k^2\right)} \times \left(\frac{n}{2}\right)^{\left(\frac{(2-n)k}{2}+(n-2)k^2\right)} \\
&\quad \times \prod_{d=\frac{n+2}{2}}^{n-1} d^{\left((2d-2n)k+(2n-2d)k^2\right)} \times n^{\left(\frac{-k}{2}+\frac{k^2}{2}\right)}.
\end{aligned}
$$

□

Corollary 3.4. *The Wiener index of $\mathscr{G}$ is given as follows;*

Case 1: *For $n = 4$,*

$$W_1(\mathscr{G}(3,k)) = -6k + 12k^2.$$

Case 2: *For $n > 4$,*

$$W_1(\mathscr{G}) = \frac{1}{4}k^2(n-1)n^2 + \frac{1}{8}k\left(-n^3 + 4n^2 - 12n\right) - \frac{1}{4}(n-4)nk.$$

Corollary 3.5. *The hyper Wiener index of $\mathscr{G}$ is given as follows;*

Case 1: *For $n = 4$,*

$$WW_1(\mathscr{G}(3,k)) = 46k^2 - 32k.$$

Case 2: *For $n > 4$,*

$$\begin{aligned} WW_1(\mathscr{G}) = {} & \frac{1}{48}k^2n(n(n(7n+8)-4)-8) + \frac{1}{48}k\left(n^2((4-5n)n+8) - 112n\right) \\ & + \frac{1}{48}(96nk - 6n(n+2)nk). \end{aligned}$$

Corollary 3.6. *The Harary index of $\mathscr{G}$ is given as follows;*

$$\begin{aligned} h_0(\mathscr{G}) = nk + \sum_{d=2,n>4}^{\frac{n-2}{2}} \frac{\left((n-2d+2)k + (2d-2)k^2\right)}{d} + \frac{\left(\frac{(2-n)k}{2} + (n-2)k^2\right)}{\frac{n}{2}} \\ + \sum_{d=\frac{n+2}{2}}^{n-1} \frac{\left((2d-2n)k + (2n-2d)k^2\right)}{d} + \frac{\left(\frac{-k}{2} + \frac{k^2}{2}\right)}{n}. \end{aligned}$$

4 Python Algorithm for Generalized Friendship Graph and Its Distance Matrix

Below is the Python code for generating the generalized friendship graph $\mathscr{G}(n,k)$ and computing its distance matrix:

```python

n = int(input('Enter the number of vertices in each cycle (n >= 3): '))
k = int(input('Enter the number of cycles sharing a common vertex (k >= 1): '))

# Step 1: Generate the generalized friendship graph
central_vertex = 0
vertices = [central_vertex]
edges = []

for i in range(k):
    cycle_vertices = range(len(vertices), len(vertices) + n - 1
    vertices.extend(cycle_vertices)
    cycle_edges = [(central_vertex, cycle_vertices[0])]
    for j in range(n - 2):
        cycle_edges.append((cycle_vertices[j], cycle_vertices[j + 1]))
    cycle_edges.append((cycle_vertices[-1], central_vertex))
    edges.extend(cycle_edges)
```

```

# Step 2: Create the graph object
G = nx.Graph()
G.add_edges_from(edges)

# Step 3: Plot the graph
pos = nx.spring_layout(G)
plt.figure(figsize=(8, 6))
nx.draw(G, pos, with_labels=True, node_color='lightblue', node_size=500, font_size=10, font_weight='bold')
plt.title(f'Generalized Friendship Graph F({k}, {n})')
plt.xlabel('X')
plt.ylabel('Y')
plt.show()

# Step 4: Compute the distance matrix
D = nx.floyd_warshall_numpy(G)  # Distance matrix
print("Distance Matrix:")
print(D)
```

Listing 1: Python algorithm for $\mathscr{G}(n, k)$.

5 Conclusion and Recommendation

In the analysis of this study, we calculated Hosoya and Harary polynomials as well as their derived indices by the modified Wiener, hyper-Wiener, Harary, multiplicative Wiener and Wiener indices. The multiplicative Wiener index demonstrated rapid growth in the graphic comparison which make it appropriate for fast-scaling analytical applications such as molecular stability prediction in chemical graph theory. Moderate growth patterns of Wiener and hyper-Wiener indices fit well in cheminformatics and network analysis.

The graphical comparisons for $n \equiv 1 \mod 2$ (Figure 5) and $n \equiv 0 \mod 2$ (Figure 6) reveal distinct growth patterns among the indices. We can achieve a quick scaling with the multiplicative Wiener index because of its rapid growth which can be applied in predictions of chemical graph theory stability and network optimization. These indices provide valuable insights into graph structures, making them suitable for a wide range of applications. Organizations can choose specific indices based on their sensitivity requirements and scalability considerations.

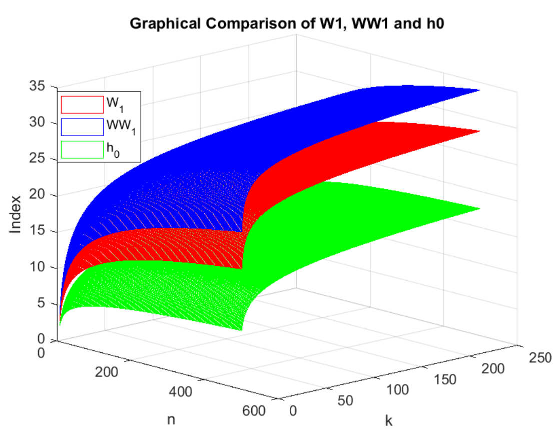

Figure 5: Indices of $\mathscr{G}(n, k)$, $1 \leq k \leq 250$, $5 \leq n \leq 500$ and $n \equiv 1 \mod 2$.

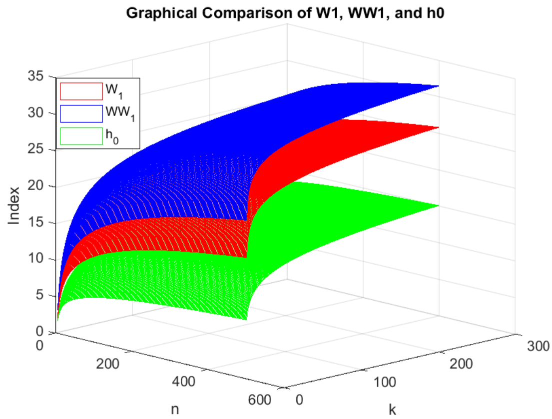

Figure 6: Indices of $\mathscr{G}(n,k)$, $1 \le k \le 250$, $4 \le n \le 500$ and $n \equiv 0 \mod 2$.

Acknowledgement We are grateful to the referee for their valuable comments, which has significantly improved the quality of our paper.

Conflicts of Interest The authors declare that they have no conflict of interest.

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